MIT OCW GR PSET 7

1 Gravitomagnetism

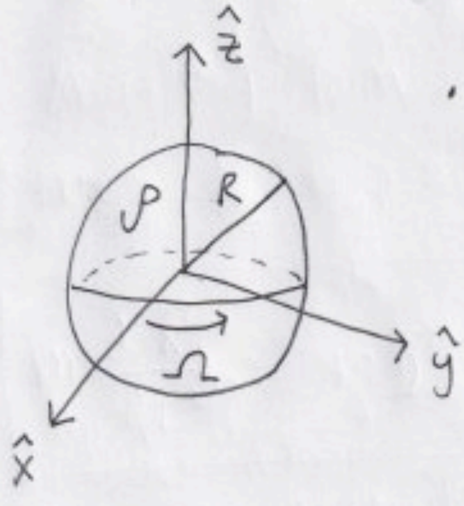
- In lecture we examined the linearized EFE in the Lorentz / Harmonic Gauge:

$$\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu} \longrightarrow \nabla^2 \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}$$

for a static source

- Now imagine the source slowly rotates and is characterized by spin spatial components S^i as well as mass M.

a · Consider the source to be a sphere of radius R & density ρ rotating about $\hat{z}$ with angular velocity Ω. Find $T_{\mu\nu}$ to 1st order in Ω:



- Assume the sphere is a collection of dust and $\gamma \approx dt/d\tau \approx 1$

$$\Longrightarrow T_{\mu\nu} = \rho u_\mu u_\nu$$

where $\vec{u} \approx (1, d\vec{x}/d\tau)$

- for ccw rotation, $V_x = -\Omega y$ and $V_y = \Omega x$ and $V_z = 0 \ldots$

$\rightarrow \vec{u} = (1, -\Omega y, \Omega x, 0)$

$$\rightarrow [T_{\nu r}] = \rho \begin{bmatrix} 1 & -\Omega y & \Omega x & 0 \\ -\Omega y & O(\Omega^2) & O(\Omega^2) & 0 \\ \Omega x & O(\Omega^2) & O(\Omega^2) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

· Let $O(\Omega^2) \rightarrow 0$

· where $x = r \sin\theta \cos\varphi$, and $y = r \sin\theta \sin\varphi$

b · solve for the Cartesian off-diagonal components h_{0x}, h_{0y}, h_{0z} where $\bar{h}_{0i} = h_{0i}$ trace reversal has no effect on off-diagonal components.

· Use the fact that:

$$h_{0i}(\vec{x}) = 4G \int \frac{T_{0i}(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|} d^3x'$$

$$\frac{1}{|\vec{x} - \vec{x}\,'|} = \frac{1}{r} + \frac{x^j x^{j'}}{r^3} + \cdots$$

· Start with h_{0x} $\longrightarrow$

$$h_{0x} = 4G\int_{V^3}\left(\frac{1}{r} + \frac{xx' + yy' + zz'}{r^3}\right)(-\Omega y'\rho)\, d^3x'$$

- Convert to spherical coords:

$$= -4G\Omega\rho\int_{V^3}\left(\frac{1}{r} + \frac{rr'}{r^3}\left(\sin\theta\sin\theta'\cos\varphi\cos\varphi' + \sin\theta\sin\theta'\sin\varphi\sin\varphi' + \cos\theta\cos\theta'\right)\right)\left(r'\sin\theta'\sin\varphi'\right) r'^2\sin\theta'\, dr'd\theta'd\varphi'$$

where $\int_{V^3} \longrightarrow \int_0^R dr' \int_0^{\pi} d\theta' \int_0^{2\pi} d\varphi'$

- The 1st part of the integrand with $\frac{1}{r}$ goes to zero since $\int_0^{2\pi} d\varphi' \sin\varphi' = -\cos\varphi'\Big|_0^{2\pi} = 0$

$$\rightarrow h_{0x} = -\frac{4G\Omega\rho R^5}{5r^2}\int_0^{2\pi} d\varphi'\int_0^{\pi} d\theta'\Big[\underbrace{\sin^3\theta'\sin\varphi'\cos\varphi'\sin\theta\cos\varphi}_{\to 0} + \sin^3\theta'\sin^2\varphi'\sin\theta\sin\varphi + \underbrace{\sin^2\theta'\cos\theta'\cos\theta}_{\to 0}\Big]$$

$$\int_0^{2\pi} d\varphi' \sin\varphi'\cos\varphi' = 0 \quad ; \quad \int_0^{\pi} d\theta' \cos\theta' = 0$$

$$\rightarrow h_{0x} = \frac{-4G\Omega\rho R^5}{5r^2}\left(\int_0^{2\pi} d\varphi' \int_0^{\pi} d\theta' \sin^3\theta' \sin^2\varphi' \sin\theta \sin\varphi\right.$$

$$\int_0^{2\pi} d\varphi' \sin^2\varphi' = \left[\frac{\varphi'}{2} - \sin(2\varphi')/4\right]\Bigg|_0^{2\pi} = \pi$$

$$\int_0^{\pi} d\theta' \sin^3\theta' = \left[-\cos(\theta') + \frac{\cos^3(\theta')}{3}\right]\Bigg|_0^{\pi}$$

$$= 1 - \frac{1}{3} + 1 - \frac{1}{3} = \frac{4}{3}$$

$$\rightarrow h_{0x} = \frac{-16\pi G\Omega\rho R^5}{15r^2} \sin\theta \sin\varphi$$

- Multiply top + bottom by "r" + plug in $y = r\sin\theta\sin\varphi$ to yield:

$$\boxed{h_{0x} = \frac{-16\pi G\Omega\rho R^5 y}{15r^3} \propto \frac{-\rho R^5 y}{r^3}}$$

- More tedious integrals + the fact that $T_{0z} = 0$ yield similarly that:

$$\boxed{h_{0y} = \frac{16\pi G\Omega\rho R^5 x}{15r^3} \propto \frac{\rho R^5 x}{r^3} \quad \text{and} \quad h_{0z} = \emptyset}$$

[c] · Using the identity $S^i = I\Omega^i$ where I is the moment of inertia[redacted] rewrite your anser in terms of S^i:

$$S^z = I\Omega^z = I\Omega, \quad \Omega^x = \Omega^y = 0$$

[redacted]

$$I_{\text{sphere}} = \frac{2}{5}MR^2$$

$$= \frac{2}{5}\left(\frac{4\pi R^3}{3}\rho\right)R^2 = \frac{8\pi\rho R^5}{15}$$

$$\rightarrow h_{ox} = \frac{-16\pi G\rho R^5}{15 r^3} \cdot [redacted]\, y \cdot \frac{15 S^z}{8\pi\rho R^5} = \frac{-2GS^z y}{r^3}$$

$$\rightarrow \boxed{h_{ox} = \frac{-2GyS^z}{r^3}}$$

similarly;

$$\boxed{h_{oy} = \frac{2GxS^z}{r^3}, \quad h_{oz} = 0}$$

· convert to spherical coordinates and find $h_{or}, h_{o\theta},$ [redacted] $h_{o\varphi}$:

Define $\vec{h}_o = [h_{ox}, h_{oy}, h_{oz}]$

$$\rightarrow \vec{h}_o = \frac{2GS^z}{r^3}[-y, x, 0]$$

$$\rightarrow \vec{h}_0 = \frac{2GS^z}{r^3} \cdot r \left[-\sin\theta \sin\varphi, \sin\theta\cos\varphi, 0 \right]$$

$$\rightarrow \vec{h}_0 = \frac{2GS^z \sin\theta}{r^2} \left[-\sin\varphi, \cos\varphi, 0 \right]$$

- According to wikipedia, in an orthonormal basis:

$$\hat{\phi} = [-\sin\varphi, \cos\varphi, 0]$$

$$\hat{r} = [\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta]$$

$$\hat{\theta} = [\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta]$$

- We want to work in a coordinate basis in which case we would append / multiply each of these unit vectors by $\sqrt{g_{\varphi\varphi}}$, $\sqrt{g_{rr}}$, and $\sqrt{g_{\theta\theta}}$ respectively.

- We can find $h_{0\theta}, h_{0\varphi}, h_{0r}$ via:

$$h_{0\theta} = \vec{h}_0 \cdot \sqrt{g_{\theta\theta}}\, \hat{\theta}$$

$$h_{0\varphi} = \vec{h}_0 \cdot \sqrt{g_{\varphi\varphi}}\, \hat{\varphi}$$

$$h_{0r} = \vec{h}_0 \cdot \sqrt{g_{rr}}\, \hat{r}$$

doing so we find that: $\rightsquigarrow$

$$h_{0\theta} = h_{0r} = 0$$

$$h_{0\varphi} = \frac{2GS^z \sin\theta}{r^2} \cdot \underbrace{\sqrt{g_{\varphi\varphi}}}_{\sqrt{r^2\sin^2\theta}} \underbrace{\left(\sin^2\varphi + \cos^2\varphi\right)}_{+1}$$

$$\rightarrow h_{0\varphi} = \frac{2GS^z \sin^2\theta}{r}$$

$$h_{0\theta} = h_{0r} = 0$$

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[2] · Comparison of linearized GR + Maxwell Theory

· Consider the line element:

$$ds^2 = -(1+2\phi)dt^2 + (1-2\phi)(dx^2+dy^2+dz^2) - 2\beta^i dx^i dt$$

i.e. the usual weak field line element with

$h^{0i} = -\beta^i$.

[a] · Show that the geodesic equation for a particle moving in this spacetime gives the following EOM:

$$m\frac{d^2\vec{x}}{d\tau^2} = m\vec{g} + m(\vec{V}\times\vec{H})$$

where $\vec{g} \equiv -\overline{\nabla}\phi$ and $\vec{H} \equiv \overline{\nabla}\times\overline{\beta}$

· First use the non-relativistic approximation $\vec{u} = (1, \underset{\sim}{V})$. Now write out the geodesic equation:

$$\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\nu\gamma}u^\nu u^\gamma$$

· Consider only spatial indices $\alpha\to i$ and plug in definition of $\vec{u}$:

$$\Rightarrow \frac{d^2x^i}{d\tau^2} = -\Gamma^\alpha_{tt}\overset{1}{u^t}\overset{1}{u^t} - \Gamma^\alpha_{ij}\underbrace{V^iV^j}_{O(V^2)\to 0}$$

$$- \Gamma^\alpha_{tj}u^tV^j - \Gamma^\alpha_{jt}V^ju^t$$

$$\Rightarrow \frac{d^2 x^i}{d\tau^2} = -\Gamma^{\alpha}_{tt} - 2\Gamma^{\alpha}_{tj} v^j$$

· Calculate Γ^i_{tt} and Γ^i_{tj} using the fact that in linearized GR:

$$\Gamma^{\nu}_{\alpha\beta} = \frac{1}{2}\eta^{\nu\sigma}\left(\partial_\alpha h_{\beta\sigma} + \partial_\beta h_{\alpha\sigma} - \partial_\sigma h_{\alpha\beta}\right)$$

(see e.g. Tapir's notes online...)

· In our case $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where

$h_{00} = -2\phi$, $h_{0i} = -\beta_i$, $h_{ij} = -2\phi$, $h_{i0} = -\beta_i$

$$\rightarrow \Gamma^i_{tt} = \left(\frac{1}{2}\eta^{i\sigma}\right)\left(\partial_t h_{t\sigma} + \partial_t h_{t\sigma} - \partial_\sigma h_{tt}\right)$$

($\partial_t h_{t\sigma} \downarrow 0$, $\partial_t h_{t\sigma} \downarrow 0$) since stationary $\Rightarrow \partial_t \rightarrow 0$

$$\rightarrow \Gamma^i_{tt} = -\frac{1}{2}\eta^{ii}\partial_i h_{tt}$$

$$= -\frac{1}{2}\eta^{ii}(-2\partial_i\phi) = \eta^{ii}\partial_i\phi$$

· Now find $\Gamma^i_{tj} = \left(\frac{1}{2}\eta^{i\sigma}\right)\left(\partial_t h_{j\sigma} + \partial_j h_{t\sigma} - \partial_\sigma h_{tj}\right)$

($\partial_t h_{j\sigma} \downarrow 0$)

$$\rightarrow \Gamma^{i}_{tj} = \left(\tfrac{1}{2}\eta^{ir}\right)\left(\partial_j h_{tr} - \partial_r h_{tj}\right)$$

$$= \tfrac{1}{2}\eta^{ii}\partial_j h_{ti} - \tfrac{1}{2}\eta^{ii}\partial_i h_{tj}$$ since η diagonal...

$$= \left(\tfrac{1}{2}\eta^{ii}\right)\left(\partial_j \beta_i - \partial_i \beta_j\right)$$

- plugging these back into our EOM we get:

$$\frac{d^2 x^i}{d\tau^2} = -\eta^{ii}\partial_i \phi + \eta^{ii}\left(\partial_i B_j - \partial_j B_i\right)V^j$$

- This is our final equation. Vectorally, this represents

$$\frac{d^2 \vec{X}}{d\tau^2} = -\bar{\nabla}\phi + \left(\bar{\nabla}\times\bar{\beta}\right)\times -\vec{V}$$

- Multiply both sides by mass "m" + plug in the definitions of $\vec{g}$ and $\vec{H}$ to yield:

$$m\frac{d^2 \vec{X}}{d\tau^2} = m\vec{g} + m\vec{V}\times\vec{H}$$ as wanted to show ✓

[b]. Show that for stationary sources (no $T_{\mu\nu}$ varies with time) the EFE can be written as:

$\rightsquigarrow$

$$\begin{cases} \bar{\nabla}\cdot\bar{g} = -4\pi G\rho \\ \bar{\nabla}\times\bar{H} = -16\pi G\bar{J} \quad \text{where } \bar{J} = \rho\bar{V} \\ \bar{\nabla}\cdot\bar{H} = 0 \\ \bar{\nabla}\times\bar{g} = 0 \end{cases}$$

where $\bar{J} = \rho\bar{V}$ is the velocity of fluid flow of the source: Start with the first:

$$\bar{\nabla}\cdot\bar{g} = -\bar{\nabla}\cdot\bar{\nabla}\phi = -\nabla^2\phi$$

Use the Newtonian approximation $\nabla^2\phi = 4\pi G\rho$

$$\Rightarrow \boxed{\bar{\nabla}\cdot\bar{g} = -4\pi G\rho \quad \checkmark}$$

- Now the second:
- Use $\Box\bar{h}_{\nu r} = \nabla^2\bar{h}_{\nu r} = -16\pi G T_{\nu r}$
- Assume a dust collection s.t. $T_{\nu r} = \rho u_\nu u_r$

$$\rightarrow \nabla^2\bar{h}_{0r} = -16\pi G T_{0r} = -16\pi G u_r \rho$$

- Narrow r down to spatial indices i:

$$\rightarrow \nabla^2\bar{h}_{0i} = -16\pi G\rho V_i$$, and now use $h_{0i} = -\beta_i$

$\rightarrow \nabla^2 \beta_i = 16 \pi G \rho V_i$

· Use the identity $\bar{\nabla} \times \bar{H} = \bar{\nabla} \times (\bar{\nabla} \times \bar{\beta})$

$= \bar{\nabla}(\bar{\nabla} \cdot \bar{\beta}) - \nabla^2 \bar{\beta}$

· And apply the Lorenz / Harmonic gauge condition for linearized GR that $\partial^\nu \bar{h}_{\nu\nu} = 0$:

$\rightarrow \partial^i \bar{h}_{i\nu} + \partial^t \bar{h}_{t\nu} = 0$

$\searrow 0$ due to stationary

· Now choose $\nu = 0$:

$\rightarrow \partial^i \bar{h}_{i0} = 0 \rightarrow \partial^i \beta_i = 0 \rightarrow \bar{\nabla} \cdot \bar{\beta} = 0$

· So we get that $\bar{\nabla} \times \bar{H} = -\nabla^2 \bar{\beta}$

$\rightarrow \bar{\nabla} \times \bar{H} = -16 \pi G \rho \bar{V}$ now insert $\bar{J} = \rho \bar{V}$

$\rightarrow$ $\boxed{\bar{\nabla} \times \bar{H} = -16 \pi G \bar{J}}$ ✓

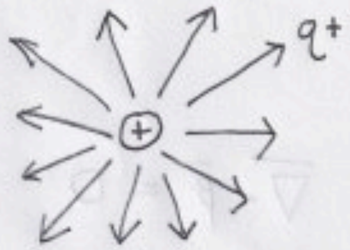
· The last two equations $\bar{\nabla} \cdot \bar{H} = 0$ & $\bar{\nabla} \times \bar{g} = 0$ follow from the vector calculus facts that

div(curl) = curl(div) = ∅

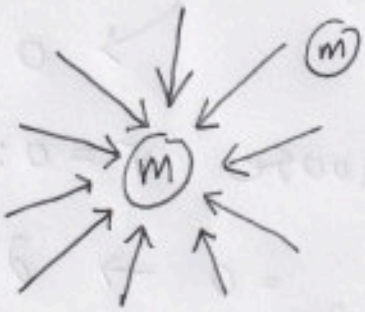
$\rightarrow$ $\boxed{\bar{\nabla} \cdot \bar{H} = \bar{\nabla} \times \bar{g} = 0}$

C . These equations bear a strong resemblance to the Maxwell equations with $\partial_t \bar{E} = \partial_t \bar{B} = 0$ except for the reversed sign in both equations and extra factor of 4 in the curl equation. Can you explain these differences?

Ans: I think the reversed sign comes down to the fact that 2 masses have gravity act as a sink whereas 2 protons have electromagnetism act as a source:



"Electric Repulsion"



"Gravitational Attraction"

. I think the extra factor of 4 in the curl equation comes down to the fact we are working in a higher dimensional space or i.e. space + time are treated more interchangeably so we have "4 space-time" coordinates rather than "3 spatial" coordinates + the "separate" flow of time...

Carroll 7.1

- Show that variation of the Lagrangian:

$$\mathcal{L} = \frac{1}{2}\Big[(\partial_\mu h^{\mu\nu})(\partial_\nu h) - (\partial_\mu h^{\rho\sigma})(\partial_\rho h^\mu_\sigma) + \frac{1}{2}\eta^{\mu\nu}(\partial_\mu h^{\rho\sigma})(\partial_\nu h_{\rho\sigma}) - \frac{1}{2}\eta^{\mu\nu}(\partial_\mu h)(\partial_\nu h)\Big]$$

Leads to the Einstein Tensor in Linear GR:

$$G_{\mu\nu} = \frac{1}{2}\left(\partial_\sigma \partial_\nu h^\sigma_\mu + \partial_\sigma \partial_\mu h^\sigma_\nu - \partial_\mu \partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu}\partial_\rho \partial_\lambda h^{\rho\lambda} + \eta_{\mu\nu}\Box h\right)$$

- We will need to use the following facts:

- $\partial_\mu(\delta g^{\mu\nu}) = \delta(\partial_\mu g^{\mu\nu})$ "variation of partial of metric equals partial of variation of metric"
- $h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}$ "indices of h raised and lowered by flat metric η"
- $\int d^4x\, \delta(\partial_\mu g^{\mu\nu})A_\lambda = -\int d^4x (\partial_\mu A_\lambda)\delta g^{\mu\nu}$ + Boundary term → 0 @ infinity..

via integration by parts...

- Begin by enforcing that: ⟶

$$0 = \delta S = \int \frac{\partial \mathcal{L}}{\partial h^{\nu\mu}} \delta h^{\nu\mu}$$

and find the variation for each of the 4 terms/summands in $\mathcal{L}$:

$$\mathcal{L}_1 \equiv \frac{1}{2}(\partial_\nu h^{\nu\mu})(\partial_\mu h)$$

$$= \frac{1}{2}(\partial_\nu h^{\nu\mu})(\partial_\mu \eta^{\alpha\beta} h_{\alpha\beta})$$

$$\rightarrow 0 = \int \delta \mathcal{L}_1 \equiv \delta S_1$$

$$= \frac{1}{2}\int \Big[(\partial_\nu \delta h^{\nu\mu})(\eta^{\alpha\beta}\partial_\mu \delta h_{\alpha\beta}) + (\partial_\nu \delta h^{\nu\mu})(\eta^{\alpha\beta}\partial_\mu h_{\alpha\beta})\Big] d^4x$$

- Apply integration by parts ....

$$= -\frac{1}{2}\int \Big[(\partial_\nu \partial_\mu h^{\nu\mu})\eta^{\alpha\beta}\delta h_{\alpha\beta} + (\partial_\nu \partial_\mu h_{\alpha\beta})\eta^{\alpha\beta}\delta h^{\nu\mu}\Big] d^4x$$

$$= -\frac{1}{2}\int \Big[\partial_\nu \partial_\sigma h^{\nu\sigma} \eta^{\alpha\beta} \eta_{\nu\alpha} \eta_{\sigma\beta} \delta h^{\nu\sigma} + \partial_\nu \partial_\sigma h \, \delta h^{\nu\sigma} \Big] d^4x$$

$$= -\frac{1}{2}\int \Big[\partial_\nu \partial_\sigma h_{\nu\sigma} \eta^{\nu\nu} \eta^{\sigma\sigma} \eta^{\alpha\beta} \eta_{\nu\alpha} \eta_{\sigma\beta} \delta h^{\nu\sigma} + \partial_\nu \partial_\sigma h \, \delta h^{\nu\sigma} \Big] d^4x$$

$$= -\frac{1}{2}\int \Big[\delta^\nu_\alpha \delta^\sigma_\beta \eta^{\alpha\beta} \partial_\nu \partial_\sigma h_{\nu\sigma} \delta h^{\nu\sigma} + (\partial_\nu \partial_\sigma h) \delta h^{\nu\sigma} \Big] d^4x$$

[redacted]

$$\rightarrow \delta S_1 = -\frac{1}{2}\int \Big[\Box h_{\nu\sigma} + \partial_\nu \partial_\sigma h \Big] \delta h^{\nu\sigma} d^4x$$

. Now onto the second terms:

$$\delta S_2 \equiv \int \delta \mathcal{L}_2 \, d^4x \quad \text{where}$$

$$\mathcal{L}_2 \equiv -\frac{1}{2}(\partial_\nu h^{\rho\sigma})(\partial_\rho h^\nu_\sigma) \longrightarrow$$

$$= -\frac{1}{2}\int \Big[\partial_\nu (\delta h^{\rho\sigma})(\eta_{\gamma\sigma}\partial_\rho h^{\nu\gamma}) + (\partial_\nu h^{\rho\sigma})(\eta_{\gamma\sigma}\partial_\rho(\delta h^{\nu\gamma}))\Big] d^4x$$

$$= \frac{+1}{2}\int \Big[\eta_{\gamma\sigma}\partial_\nu\partial_\rho h^{\nu\gamma}\delta h^{\rho\sigma} + \partial_\nu\partial_\rho h^{\rho\sigma}\eta_{\gamma\sigma}\delta h^{\nu\gamma}\Big] d^4x$$

↑ let $\rho \leftrightarrow \sigma$, $\sigma \leftrightarrow \rho$

$$= \frac{+1}{2}\int \Big[\eta_{\gamma\sigma}\partial_\nu\partial_\rho h^{\nu\gamma}\delta h^{\rho\sigma} + \big[\partial_\nu\partial_\sigma h^{\sigma}_{\ \gamma}\delta h^{\nu\gamma}\big] d^4x$$

→ Let $\rho \leftrightarrow \gamma$, $\sigma \leftrightarrow \nu$

$$= \frac{+1}{2}\int \Big[\partial_\sigma\partial_\gamma \eta_{\rho\nu} h^{\sigma\rho}\delta h^{\nu\gamma} + \partial_\nu\partial_\sigma h^{\sigma}_{\ \gamma}\delta h^{\nu\gamma}\Big] d^4x$$

$$= \frac{+1}{2}\int \Big[\partial_\sigma\partial_\gamma h^{\sigma}_{\ \nu} + \partial_\nu\partial_\sigma h^{\sigma}_{\ \gamma}\Big]\delta h^{\nu\gamma} d^4x \equiv \delta S_2$$

- Now onto the third piece of the overall Lagrangian:

$$\mathcal{L}_3 \equiv \frac{1}{4}\eta^{\nu\gamma}(\partial_\nu h^{\rho\sigma})(\partial_\gamma h_{\rho\sigma}) \rightsquigarrow$$

$$\delta S_3 \equiv \int \delta \mathcal{L}_3 \, d^4x$$

$$= \frac{1}{4}\int \Big[\eta^{\nu r} \partial_\nu (\delta h^{\rho\sigma}) (\partial_r h_{\rho\sigma}) + \eta^{\nu r} (\partial_\nu h^{\rho\sigma}) \partial_r (\delta h_{\rho\sigma}) \Big] d^4x$$

$$= -\frac{1}{4}\int \Big[\eta^{\nu r} (\partial_\nu \partial_r h_{\rho\sigma}) \delta h^{\rho\sigma} + \eta^{\nu r} \partial_\nu \partial_r h^{\rho\sigma} \delta h_{\rho\sigma} \Big] d^4x$$

flip lower ↔ upper

$$= -\frac{1}{2}\int \eta^{\nu r} \partial_\nu \partial_r h_{\rho\sigma} \, \delta h^{\rho\sigma} \, d^4x$$

$$\eta^{\nu r}\partial_\nu\partial_r \rightarrow \eta_{\nu r}\partial^\nu \partial^r \rightarrow \eta_{\nu r}\eta^{\nu\rho}\eta^{r\lambda}\partial_\rho\partial_\lambda$$

$$\delta h^{\rho\sigma} \rightarrow \eta^\rho_\nu \eta^\sigma_r \delta h^{\nu r}$$

$$h_{\rho\sigma} \rightarrow h^{\rho\lambda}\eta_{\lambda\rho}\eta_{\rho\sigma}$$

$$= -\frac{1}{2}\int \eta^{\nu\rho}\eta^{r\lambda}\eta_{\lambda\rho}\eta_{\rho\sigma} \, \eta_{\nu r}\partial_\rho\partial_\lambda h^{\rho\lambda} \, \eta^\rho_\nu \eta^\sigma_r \delta h^{\nu r} \cdot d^4x$$

$$\rightarrow \eta^{\nu\rho}\eta^{r\lambda}\eta_{\lambda\rho}\eta_{\rho\sigma}\eta^\rho_\nu\eta^\sigma_r$$

$$= \delta^{\nu}_{\lambda}\, \eta^{\mu\lambda}\, \eta_{\rho\sigma}\, \eta^{\rho}_{\nu}\, \eta^{\sigma}_{\mu}$$

$\eta_{\rho\sigma}\eta^{\rho}_{\nu} \to \delta_{\sigma\nu}$

$\eta^{\mu\lambda}\eta^{\sigma}_{\mu} \to \delta^{\lambda\sigma}$

$$= \delta^{\nu}_{\lambda}\, \delta^{\lambda\sigma}\, \delta_{\sigma\nu} = \delta^{\nu}_{\lambda}\, \delta^{\lambda}_{\nu} = \delta^{\lambda}_{\lambda} \to \text{"Identity Matrix"}$$

· <u>So overall</u> ...

$$\delta S_3 = -\frac{1}{2}\int \eta_{\nu\mu}\, \partial_\rho \partial_\lambda h^{\rho\lambda}\, \delta h^{\nu\mu}\, d^4x$$

. Now the forth + final piece of the Lagrangian:

$$\mathcal{L}_4 \equiv -\frac{1}{4}\eta^{\nu\mu}(\partial_\nu h)(\partial_\mu h)$$

$$\delta S_4 \equiv \int \delta \mathcal{L}_4\, d^4x$$

$$= -\frac{1}{4}\int \Big[\eta^{\nu\mu}\partial_\nu(\delta h_{\alpha\beta}\,\eta^{\alpha\beta})(\partial_\mu h_{\gamma\omega}\,\eta^{\gamma\omega}) + \eta^{\nu\mu}(\partial_\nu h_{\alpha\beta}\,\eta^{\alpha\beta})\,\partial_\mu(\delta h_{\gamma\omega}\,\eta^{\gamma\omega}) \Big] d^4x$$

$$= +\frac{1}{4}\int \left[\eta^{\mu\nu}\eta^{\gamma\omega}\eta^{\alpha\beta}(\partial_\mu \partial_\nu h_{\gamma\omega})\,\delta h_{\alpha\beta} + \eta^{\mu\nu}\eta^{\gamma\omega}\eta^{\alpha\beta}(\partial_\mu \partial_\nu h_{\alpha\beta})\,\delta h_{\gamma\omega} \right] d^4x$$

h

h

② flip lower to upper
+ relabel dummies to $\mu\nu$

$$= +\frac{1}{4}\int 2\eta^{\mu\nu}\partial_\mu\partial_\nu h \, \eta_{\mu\nu}\,\delta h^{\mu\nu}\, d^4x$$

$$= +\frac{1}{2}\int \eta_{\mu\nu}\,\Box h\,\delta h^{\mu\nu}\, d^4x \equiv \delta S_4$$

· <u>Now</u> we get:

$$\delta S_{tot} = \delta S_1 + \delta S_2 + \delta S_3 + \delta S_4 = 0$$

$$\rightarrow \frac{1}{2}\Big(-\Box h_{\mu\nu} - \partial_\mu\partial_\nu h + \partial_\sigma \partial_\nu h^\sigma_\mu + \partial_\sigma\partial_\mu h^\sigma_\nu - \eta_{\mu\nu}\partial_\rho\partial_\lambda h^{\rho\lambda} + \eta_{\mu\nu}\Box h \Big) = 0 \quad (\star)$$

- Equation ⊛ exactly matches the linearized Einstein tensor given by Carroll & we have since found the EFE in Linear GR with no sources (i.e. $T_{\mu\nu} \to 0$):

$$\to G_{\mu\nu} = 0 \quad ✓\!\!✓$$

Q.E.D.

Carroll 7.4

- Show that the Harmonic gauge $\Box x^{\nu} = 0$ is equivalent to the Lorentz gauge $\partial_{\nu} \bar{h}^{\mu\nu} = 0$:

  - Let's first simplify $\Box x^{\nu} = 0$:

  - In flat-spacetime, $\Box \equiv \eta^{\alpha\beta} \partial_{\alpha} \partial_{\beta}$; however, more generally, $\Box \equiv g^{\alpha\beta} \nabla_{\alpha} \nabla_{\beta}$. We have to use this more general form in our derivation...

- Also note that x^{ν} is not a vector, but just a single coordinate component so the first ∇_{β} reduces to $\nabla_{\beta} x^{\nu} = \partial_{\beta} x^{\nu} = \delta^{\nu}_{\beta}$

$$\to \Box x^{\nu} = 0$$

$$\to g^{\alpha\beta} \nabla_{\alpha} \left(\delta^{\nu}_{\beta} \right) = 0$$

$$\rightarrow g^{\alpha\beta}\Big(\underbrace{\partial_\alpha \delta_\beta^{\nu}}_{\emptyset} + \Gamma^{\nu}_{\alpha\beta}\underbrace{\delta_\beta^{\beta}}_{1}\Big) = 0$$

$$\rightarrow g^{\alpha\beta}\Gamma^{\nu}_{\alpha\beta} = 0 \quad (\star)$$

• We will need to use this relationship later... It is an equivalent form of $\Box x^\nu = 0$

• Now use the fact that we have metric compatibility + take the divergence of the metric:

$$\nabla_\nu g^{\nu\sigma} = 0$$

$$\rightarrow \partial_\nu g^{\nu\sigma} + \Gamma^{\nu}_{\nu\alpha} g^{\alpha\sigma} + \underbrace{\Gamma^{\sigma}_{\alpha\nu} g^{\alpha\nu}}_{\emptyset \text{ by } (\star)} = 0$$

$$\rightarrow \partial_\nu g^{\nu\sigma} + \Gamma^{\nu}_{\nu\alpha} g^{\alpha\sigma} = 0$$

• relabel [redacted] dummies $\nu \rightarrow \sigma$

$$\rightarrow \partial_\nu g^{\nu\sigma} + \underbrace{\Gamma^{\sigma}_{\alpha\sigma} g^{\alpha\sigma}}_{0 \text{ by } (\star)} = 0$$

$$\rightarrow \partial_\nu g^{\nu\sigma} = 0 \quad (*)$$

• So we have show that given metric compatibility + the Harmonic gauge $\Box x^{\mu} = 0$ not only does $\nabla_{\nu} g^{\mu\nu} = 0$ but also $\partial_{\nu} g^{\mu\nu} = 0$

• Now we will express $g^{\mu\nu}$ in terms of $\bar{h}^{\mu\nu}$ to recover the Lorentz gauge condition:

• We know from Carroll's text that $\bar{h}_{\mu\nu}$ is defined s.t. its trace $\eta^{\mu\nu}\bar{h}_{\mu\nu}$ equals the negative trace "h" of $h_{\mu\nu}$. We will find $\bar{h}^{\mu\nu}$ [redacted] by setting its trace to $-h$ as well:

$$\eta_{\mu\nu}\bar{h}^{\mu\nu} = \bar{h} = -\eta_{\mu\nu}h^{\mu\nu} = -h$$

$$\rightarrow \bar{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h$$

• Now use the facts that:

$$\left\{ \begin{array}{l} g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \\ g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \end{array} \right\}$$

$$\rightarrow \bar{h}^{\mu\nu} = \eta^{\mu\nu} - g^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\eta_{\alpha\beta}(\eta^{\alpha\beta} - g^{\alpha\beta})$$

$$\rightarrow \bar{h}^{\nu\sigma} = \eta^{\nu\sigma} - g^{\nu\sigma} - 2\eta^{\nu\sigma} + \frac{1}{2}\eta^{\nu\sigma}\eta_{\alpha\beta}g^{\alpha\beta}$$

change dummies to $\nu\sigma$

$$\rightarrow \bar{h}^{\nu\sigma} = -g^{\nu\sigma} - \eta^{\nu\sigma} + 2g^{\nu\sigma}$$

$$\rightarrow \bar{h}^{\nu\sigma} = g^{\nu\sigma} - \eta^{\nu\sigma}$$

- Now take partial w.r.t. ν:

$$\partial_\nu \bar{h}^{\nu\sigma} = \partial_\nu g^{\nu\sigma} - \partial_\nu \eta^{\nu\sigma} \rightarrow 0$$

$$\rightarrow \partial_\nu \bar{h}^{\nu\sigma} = \partial_\nu g^{\nu\sigma} = 0 \quad \text{by relation } (*)$$

- so in the end we recover $\boxed{\partial_\nu \bar{h}^{\nu\sigma} = 0}$

as we wanted to show W Q.E.D.

Carroll Chapter 7, Part 3

- Fermat's principle states that a light ray moves along a path of least time. For a medium with refractive index $n(\vec{x})$ this is equivalent to extremizing the time:

$\longrightarrow$

$$t = \int n(\vec{x}) \, [\delta_{ij} \, dx^i dx^j]^{1/2}$$ along the path.

· Show that Fermat's principle with refractive index $n = 1 - 2\phi$ leads to the correct EOM for a photon in a spacetime perturbed by a Newtonian potential:

· A photon is lightlike/null which means its invariant interval obeys the following rule:

$$0 = ds^2 = g_{\mu\nu} \, dx^\mu dx^\nu = g_{\mu\nu} \frac{dx^\mu}{dt} \cdot \frac{dx^\nu}{dt} \quad (*)$$

· Break this up explicitly:

$$0 = g_{tt} \left(\frac{dt}{dt}\right)^2 + g_{ij} \frac{dx^i}{dt} \cdot \frac{dx^j}{dt}$$

$$\rightarrow -g_{tt} = (\eta_{ij} - 2\phi \, \delta_{ij}) \left(\frac{dx^i}{dt} \cdot \frac{dx^j}{dt}\right)$$

$$\rightarrow -g_{tt} = (1 - 2\phi) \, \delta_{ij} \frac{dx^i}{dt} \cdot \frac{dx^j}{dt}$$

$$\rightarrow -g_{tt} = (1 - 2\phi) \frac{ds^2}{dt^2}$$

$$\rightarrow -(-1 - 2\phi) = (1 - 2\phi) \left(\frac{ds}{dt}\right)^2$$

$$\rightarrow \frac{1+2\phi}{1-2\phi} = \left(\frac{ds}{dt}\right)^2$$

$$\rightarrow \left(\frac{ds}{dt}\right)^2 \approx (1+2\phi)^2 \quad \text{since } \phi \ll 1 \Rightarrow \phi^2 \approx 0$$

$$\rightarrow (1+2\phi) = ds/dt$$

$$\rightarrow \int \frac{ds}{1+2\phi} = \int dt$$

$$\rightarrow \int (1-2\phi)\, ds = t$$

$$\rightarrow \boxed{\int n(\vec{x}) \left[\delta_{ij}\, dx^i dx^j\right]^{1/2} = t} \quad ✓ \; (\star)$$

• So (*) and (⋆) are logically equivalent. Starting from the EOM of a photon we can derive the Fermat integral or we could equivalently work our way backward as well... ✓